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THE LICHNEROWICZ CONJECTURE ON HARMONIC MANIFOLDS

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0. Introduction

The theory of harmonic manifolds has a relatively long history. It started with a work of H. S. Ruse in 1930, who made an attempt to find a solution for the equation $\Delta f = 0$ on a general Riemannian manifold which depends only on the geodesics distance $r(x, \cdot)$. His main aim was to use these functions to develop harmonic analysis on Riemannian manifolds similar to the euclidean case.

It turned out that such radial harmonic functions exist only in very special cases, namely, in the cases where the density function $\omega_p := \sqrt{|\det g_{ij}|}$ in the normal coordinate neighborhood $\{x^1, \cdots, x^n\}_p$ around each point p depends only on $r(p, \cdot)$. From the well-known symmetry $\omega_p(q) = \omega_q(p)$ it can be easily seen that this is the case if and only if the function $\omega_p(q)$ is of the form

$$\omega_p(q) = \phi(r(p,q)); \qquad \phi\colon \mathbf{R}_+ \to \mathbf{R},$$

where $\mathbf{R}_+$ is the set of all positive real numbers, and $\mathbf{R}$ is the set of all real numbers. A Riemannian manifold was defined to be harmonic precisely when its density function $\omega_p(q)$ satisfies this radial property.

For a precise formulation one can introduce the notions of *global, local,* and *infinitesimal harmonicity* [5]. The global (respectively, local) harmonicity refers to the case where the above radial property of the density function is global (respectively, local). For infinitesimal harmonicity we assume only that the derivatives $\nabla^{(k)}_{\xi_p \cdots \xi_p} \omega_p$ with respect to the unit vectors $\xi_p \in T_p(M^n)$ define constant functions on the manifold. These notions are obviously equivalent for analytic Riemannian manifolds [5].

The derivatives $\nabla^{(k)}_{\xi_p \cdots \xi_p} \omega_p$ can be expressed with the help of the curvature tensor and its covariant derivatives. For example, we have

$$\nabla^{(2)}_{\xi_p \xi_p} \omega_p = -\frac{1}{3} R(\xi_p, \xi_p),$$

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where $R(X, Y)$ is the Ricci curvature, so the harmonic manifolds of any type are Einstein manifolds. On the other hand, any Einstein metric is analytic in the harmonic and normal coordinates by the Kazdan-De Turck theorem [6]. Thus we get

The global, local and infinitesimal harmonicity are equivalent properties.

We mention that in another paper we shall prove that also these spaces which satisfy the Legendre curvature condition $R_{ij|k} + R_{jk|i} + R_{ki|j} = 0$ are real analytic. It follows that all the commutative spaces and D'Atri spaces are analytic.

An interesting equivalent formulation of harmonicity was found by Willmore [44]:

A Riemannian space is harmonic if and only if for any harmonic function u the classical mean-value theorem

$$u(p) = \frac{1}{\int_{S_{p;r}} dS_{p;r}(x)} \int_{S_{p;r}} u dS_{p;r}(x)$$

holds, where $dS_{p;r}(x)$ means the induced measure on the geodesic sphere $S_{p;r}$ with the center p and radius r.

Any two-point homogeneous manifold is obviously harmonic. The main problem about the harmonic manifolds was to prove *the Lichnerowicz conjecture* [23] asserting the converse statement: *Any harmonic manifold is two-point homogeneous.*

The conjecture has been proved so far only for dimensions ≤ 4 [23], [43], [5]. All these solutions use the dimensionality very heavily, and did not give any hope for higher dimensions. In higher dimensions, only partial results were proved using an additional strong assumption. One such theorem is the following.

Any locally symmetric harmonic manifold is two-point homogeneous.

The harmonic spaces were investigated from a local point of view in most cases. Among the few global investigations we mention the *Allamigeon theorem* [2] and the "*nice imbedding theorem*" of Besse [5]. The first theorem asserts that any complete simply connected harmonic manifold is diffeomorphic either to a Euclidean d-space $\mathbf{R}^d$ or to a Blaschke manifold, which has simple closed geodesics with the same length. In Besse's theorem an isometric imbedding ϕ: $M^n \to \mathbf{R}^d$ is constructed for compact simply connected harmonic spaces such that $\phi(M^n)$ is minimal in a certain sphere, and, furthermore, all the geodesics are congruent screw lines in $\mathbf{R}^d$. Both theorems will be used in the present paper.

Besides giving a general consideration our aim is to prove the conjecture for simply connected compact harmonic spaces. Using the universal covering spaces, *this proof gives a proof of the Lichnerowicz conjecture for the compact* (infinitesimal, local or global) *harmonic manifolds which have a finite fundamental group* (and hence a compact universal covering space).

It should be remarked that this proof also gives a new direct proof to the following fact for the compact case:

Any two-point homogeneous space with finite fundamental group is a symmetric space.

The above result was proved without using Wang's classification by S. Helgason for the noncompact case and by H. Matsumoto for the compact case. Using universal covering spaces, the combination of our proof with Helgason's gives:

Any two-point homogeneous space is symmetric.

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1. The basic commutativity in harmonic spaces

For the sake of simplicity we investigate in this paper simply connected complete Riemannian manifolds (M^n, g), where the metric g is assumed to be positive definite.

Let $(x_1, \cdots, x_n)_p$ be a normal coordinate neighborhood around a point $p \in M^n$. The function

$$\omega_p := \omega\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right)$$

stands for the volume density in the space. We introduce also the polar coordinate neighborhood (r_p, φ) around p, where $r_p(q) = r(p,q)$ denotes the geodesic distance between p and q, and φ represents the points of the unit sphere in the tangent space $T_p(M^n)$ at p. In this system the function ω_p can be written in the form $\omega_p(r, \varphi)$, and the density with respect to (r_p, φ) is

$$\Theta_p := r^{n-1}\omega_p.$$

A Riemannian manifold is said to be *harmonic* if the density Θ_p is a radial or spherical symmetric function around any point $p \in M^n$, i.e., it depends only on the variable r and thus can be written as $\Theta_p(r)$.

From the symmetry $\Theta_p(r_p(q)) = \Theta_q(r_q(p))$ it follows easily that the functions $\Theta_p(r)$, $p \in M^n$, are also independent of the points $p \in M^n$ in a harmonic space [5].

Now let $S_{p,R}$ be a geodesic sphere around p with radius R whose Minkowskian mean curvature is denoted by $\sigma_p(R, \varphi)$. The formula

$$(1.1) \qquad \sigma_p(R,\varphi) = \frac{\Theta_p'(R,\varphi)}{\Theta_p(R,\varphi)} = \frac{n-1}{R} + \frac{\omega'(R,\varphi)}{\omega(R,\varphi)} = -(\Delta r_p)(R,\varphi)$$

is rather well known [5], where the comma means the derivation with respect to radial direction, and $\Delta := -\nabla_i \nabla^i$ is the Laplace operator in the space.

From this formula it follows that a Riemannian manifold is harmonic iff the mean curvature function $\sigma_p(R, \cdot)$ is a radial function of the form $\sigma_p(r, \varphi) = \sigma(r(p, \cdot))$. The statement can be proved by solving the equation

$$\sigma(r) - (n-1)/r = \omega'(r)/\omega(r)$$

with the initial condition $\omega(0) = 1$.

We also mention here another connection between the Laplace operator Δ and the mean curvature function $\sigma_p(r, \varphi)$. Let $\tilde{\nabla}$ (resp. $\tilde{\Delta}$) be the covariant derivative (resp. the Laplace operator) of a geodesics sphere $S_{p,r}$ whose second fundamental form is denoted by $M_{p,r}(X, Y)$. Then the formula

$$\nabla^2 f(X,X) = X \cdot X(f) - (\nabla_X X) \cdot (f) = \tilde{\nabla}^2 f(X,X) + M(X,X) f'$$

holds for a function f in M^n and a vector field X tangent to $S_{p,r}$. Thus

$$(1.2) \qquad \Delta f = \tilde{\Delta} f - f'' - \sigma_p(r,\varphi) f',$$

and therefore the action of Δ on a radial function f (around p) is

$$(1.3) \qquad \Delta f := -f'' - \sigma_p(r,\varphi) f'.$$

We introduce also the so-called *averaging operators* A_p, $p \in M^n$, which play a very important role in the following discussions.

Let f be a smooth function of M^n. Then the averaged function $A_p(f)$ is defined as a radial function around p whose values are, at the points of a geodesic sphere $S_{p,r}$, just the average of f on $S_{p,r}$, i.e.,

$$(1.4) \qquad A_p(f)(r) = \frac{1}{\int \Theta_p(r,\varphi)\, d\varphi} \int f(r,\varphi) \Theta_p(r,\varphi)\, d\varphi.$$

The function $A_p(f)$ is defined only locally, namely for the small values of r which are less than the injectivity radius of M^n at $p \in M^n$. On the other hand $A_p(f)$ is globally defined for any $p \in M^n$ if the space is a compact Blaschke manifold (i.e., if the cut values for the space M^n are equal at any tangent space $T_p(M^n)$) or a noncompact complete manifold with an infinite injectivity radius. In the last case M^n is diffeomorphic to $\mathbb{R}^n$ by the exponential map. We call these spaces *globally averageable spaces.* The compact Blaschke manifolds have simply closed geodesics with a common length $2L$ such that the geodesics, starting from a point m, intersect the cut locus at the distance L orthogonally (see Corollary 5.42 and Proposition 7.9 in [5]). From this statement we get easily that the averaged function $A_p(f)$ of a function f of class C^k is a globally defined function of class C^k in any globally averageable space.

By the Allamigeon theorem any simply connected complete harmonic manifold is globally averageable space.*

Lemma 1.1 (*Basic commutativity in harmonic spaces*). *A Riemannian manifold (M^n, g) is harmonic if and only if the Laplace operator Δ commutes with the averaging operators, i.e.,*

$$A_p(\Delta f) = \Delta A_p(f) \tag{1.5}$$

yields for any smooth function f and $p \in M^n$.

Proof. If M^n is harmonic, then by (1.2), (1.3) and the Stokes theorem we get

$$\begin{aligned} A_p(\Delta f) &= A_p(\tilde{\Delta} f) - A_p(f'') - A_p(\sigma_p(r) f') \\ &= -(A_p(f))'' - \sigma_p(r)(A_p(f))' = \Delta A_p(f), \end{aligned} \tag{1.6}$$

which proves the commutativity in harmonic manifolds.

Conversely, if the commutativity

$$A_p(\Delta f) = \Delta A_p(f) = -(A_p(f))'' - \sigma_p(r, \varphi)(A_p(f))' \tag{1.7}$$

holds, then the mean curvature

$$\sigma_p = [-(A_p(f))'' - A_p(\Delta f)]/(A_p f)' \tag{1.8}$$

is a radial function, and the space is harmonic. q.e.d.

The above characterization of harmonic manifolds has several advantages. To make these perfectly clear we investigate here also the heat kernel on these manifolds.

At first we consider a compact Riemannian manifold M^n, and several investigations for the noncompact case will be given later.

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* On the cut locus C_p of a Blaschke manifold the averaging A_p is defined by limit. Around any point $P \in C_p$ the manifold is a (topologic) Cartesian product: $U \times V$, where $V \subset C_p$. Furthermore the submanifold U is described (locally) by the geodesics intersecting C_p at P orthogonally. For harmonic manifolds the $A_p(f)$ is a radial function on U around P which has the radial even derivatives $(1/\Omega_{n-1}) \int (d^{2e} f/dr^{2e})(L, \varphi)\, d\varphi$ and vanishing odd derivatives at P.

The heat operator of M^n is defined by

$$L := \Delta + \partial/\partial t, \tag{1.9}$$

and a solution of the heat equation $L(u) = 0$ is called a heat flow. The solutions of this equation can be determined by the heat kernel $H_t(x,y)$. This kernel function is defined on $M^n \times M^n \times \mathbf{R}_+$ and is characterized by the following properties:

(1.10)

(1) It is of class C^1 with respect to the variable t, and of class C^2 with respect to the other variables.
(2) $L_y H_t(x,y) = 0$ for any fixed point $x \in M^n$.
(3) Set $H_t^x(y)$ for the function $y \to H_t(x,y)$, then
$$\lim_{t\to+0} H_t^x = \delta_x \quad \text{(Dirac } \delta\text{-function)}$$
is satisfied for any $x \in M^n$.

The *existence* of such a kernel is assured by well-known constructions [4]. The usual simple proof of the *uniqueness* is as follows.

Let $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the (discrete!) spectrum of the Laplace operator Δ. Furthermore, let $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ be the corresponding orthonormal set of eigenfunctions forming a basis in the L^2 function space of M^n. The series

$$H_t^x(y) = \sum f_i(x,t)\varphi_i(y)$$

stands for the L^2 expansion of $H_t^x(y)$ for the fixed points t and x, and thus

$$f_i(x,t) = \int H_t(x,y)\varphi_i(y)\,dy. \tag{1.11}$$

By properties (1) and (2) we get

$$\partial f_i/\partial t = -\lambda_i f_i, \tag{1.12}$$

and therefore by property (3),

$$f_i(x,t) = e^{-\lambda_i t}\varphi_i(x), \qquad H_t(x,y) = \sum e^{-\lambda_i t}\varphi_i(x)\varphi_i(y), \tag{1.13}$$

which proves the uniqueness of the kernel $H_t(x,y)$ on compact Riemannian manifolds.

The series (1.13) is absolutely convergent by the Parseval formula for the integral

$$\int H_{t/2}(x,z)H_{t/2}(y,z)\,dz. \tag{1.14}$$

Also the series

$$\sum e^{-\lambda_i t} = \int H_t(x,x)\,dx, \qquad t > 0, \tag{1.15}$$

is convergent by the Beppo-Levi theorem.

The heat kernel $H_t(x,y)$ is used for the solution of the heat equation $L(u) = 0$ with the initial condition $u(x,0) = u_0(0)$ by the formula

$$u(x,t) = \int H_t(x,y)u_0(y)\,dy.$$

The situation is much more complicated regarding the heat kernel of a noncompact Riemannian space, as the Laplace operator does not have a discrete spectrum in these cases and we cannot use an orthonormal basis of the eigenfunctions. On the other hand we can derive several heat kernels in such spaces because of the determined boundary conditions. Very recent results refer to the existence and uniqueness of the heat kernel of a complete noncompact Riemannian manifold which vanishes at infinity [47], [14], [38]. In the following we use this heat kernel in the case of complete noncompact manifolds. We mention that the assumption on the Ricci curvature in Yau's theorem is trivially satisfied here, since now the manifold is Einsteinian with constant norm $||R||$ of the curvature tensor.

A complete Riemannian manifold without boundary is said to be *strongly harmonic* if the heat kernel $H_t(x,y)$ is a function only of t and the distance $r(x,y)$, i.e., it is of the form $H_t(x,y) = H_t(r(x,y))$.

In this case the functions $H_t^x = H_t(x,\cdot)$ are radial functions around x. Obviously this weaker property characterizes strong harmonicity, taking into account the symmetry $H_t(x,y) = H_t(y,x)$.

Any strongly harmonic manifold is harmonic (see [5, p. 172]), as can be seen from

$$\Delta_y H_t^x = -H_t^{x''} - \frac{\Theta_x'}{\Theta_x} H_t^{x'} = -\frac{\partial H}{\partial t}. \tag{1.16}$$

From this equation we get that the mean curvature $\sigma_x = \Theta_x'/\Theta_x$ is also a radial function, and so the space is harmonic.

The converse statement is also true for simply connected and complete harmonic manifolds, as was proven by D. Michel [26] using the technical method of Brownian motion. Since this theorem immediately follows from our Basic Commutativity (1.5), we give the complete proof here.

Theorem 1.1. *In the class of simply connected complete Riemannian manifolds, harmonicity and strong harmonicity are equivalent properties.*

Proof. We have to prove only that harmonicity implies strong harmonicity.

The simply connected complete harmonic manifolds are globally averageable spaces by the Allamigeon theorem. Therefore the averaged kernel

$$\tilde{H}_t(x,y) := (A_x H_t^x(y)) \tag{1.17}$$

is a globally defined smooth function which obviously satisfies (3) of (1.10). The equation $L_y\tilde{H}_t(x,y) = 0$ follows immediately from the Basic Commutativity (1.5), and $\tilde{H}_t(x,y) = H_t(x,y)$ from the uniqueness of the heat kernel. This proves the radial symmetry of the heat kernel, which is just the statement of the Theorem.

2. The analysis of radial functions in harmonic spaces

Any function $f\colon \mathbf{R}_+ \to \mathbf{R}$ generates a radial function f_x around a point $x \in M_n$ defined by $f_x(y) := f(r(x,y))$, where $r(x,y)$ means the geodesic distance between the points $x, y \in M_n$. This function is well defined only for the points y for which $r(x,y)$ is less than the injectivity radius at x. The *supporting radius* of the function f is defined by the infimum of the values $R \in \mathbb{R}_+$ for which $f([R,\infty)) = 0$ holds. If this radius is less than the injectivity radius at x, then the function f_x is globally defined on M^n.

Now let (M^n, g) be a harmonic manifold with the density function Θ_p. We consider also an eigenfunction φ of the Laplacian Δ with the eigenvalue $\lambda > 0$. From the Basic Commutativity $A_x\Delta\varphi = \Delta A_x\varphi$ it follows that the radial function $(A_x\varphi)(r)$ is an eigenfunction with the eigenvalue λ again. Thus the function $z(r) := (A_x\varphi)(r)$ is the solution of the differential equation

$$z'' + \sigma(r)z' + \lambda z = 0 \tag{2.1}$$

with the initial conditions $z(0) = \varphi(x)$ and $z'(0) = 0$. One of the difficulties with this equation is that $\sigma(r)$ has infinite value at $r = 0$; more precisely, it is of the form $\sigma(r) = \sigma^*(r)/r$ with $\sigma^*(0) = n - 1$. The following lemma plays an important role throughout the whole paper; it can be found also in [3] with a different proof.

Lemma 2.1. *The differential equation*

$$z'' + \sigma(r)z' + \lambda z = 0,$$

where $\lambda > 0$ and $\sigma(r) > 0$ near zero, has only one solution with the initial conditions $z(0) = 1$, $z'(0) = 0$.

Proof. For this uniqueness it is enough to prove that the only solution of (2.1) with $z(0) = 0$, $z'(0) = 0$ is the zero function. Now let z be such a solution. So by multiplication with z' we get

$$z''z' + \sigma(z')^2 + \lambda z z' = 0; \qquad \frac{((z')^2)'}{2} + \sigma(z')^2 + \frac{(\lambda z^2)'}{2} = 0. \tag{2.2}$$

By introducing the function

$$v = \tfrac{1}{2}((z')^2 + \lambda z^2) \geq 0, \tag{2.3}$$

we reduce the second equation in (2.2) to

$$v' = -(z')^2\sigma \leq 0, \tag{2.4}$$

which means that the function v is nonincreasing in a neighborhood of $r = 0$. Since $v \geq 0$ and $v(0) = 0$, we have $v = 0$, $z' = 0$, and $z = 0$ in this neighborhood. Thus $z = 0$ everywhere by the Picard-Lindelöff theorem, and the lemma is proved completely.

If $\varphi(x) \neq 0$ at a point $x \in M^n$ for the eigenfunction φ with the eigenvalue λ, then the function

$$\varphi^\lambda(r) := \frac{1}{\varphi(x)}(A_x\varphi)(r) \tag{2.5}$$

satisfies (2.1) with the initial conditions $\varphi^\lambda(0) = 1$ and $(\varphi^\lambda)'(0) = 0$, so $\varphi^\lambda(r)$ is uniquely determined and independent of the choice of the point x. Furthermore

$$(A_x\varphi)(r) = \varphi(x)\varphi_x^\lambda(r) \tag{2.6}$$

also holds. This formula can be considered as a generalization of the mean value theorem for the eigenfunctions of the Laplacian in harmonic manifolds, since for harmonic functions φ we have $\varphi_x^0(r) = 1$.

Formula (2.6) means also that for a fixed point $x \in M^n$ the averaging operator A_x projects the space of eigenfunctions with a common eigenvalue λ onto a one-dimensional function space. Also the formula

$$A_y(A_x\varphi)(r) = (A_x\varphi)(y)\varphi_y^\lambda(r) \tag{2.7}$$

follows immediately from (2.6).

Next we give a new characterization of harmonicity.

If $H(x,y)$ and $G(x,y)$ are two kernel functions on a Riemannian manifold such that for any x the functions $H_x(\cdot) := H(x,\cdot)$, $G^x(\cdot) := G(\cdot,x)$ are L^2-functions, then the convolution $H * G$ is defined as usual by

$$H * G(x,y) := \int H(x,z)G(z,y)\,dz. \tag{2.8}$$

In the following Proposition we investigate the convolution of two radial kernel functions of the form $H(r(x,y))$; $G(r(x,y))$, where the functions H; G: $\mathbf{R}_+ \to \mathbf{R}$ are of compact support.

Proposition 2.1. *A simply connected complete Riemannian manifold is harmonic if and only if the convolution of two radial kernel functions is a radial kernel function again.*

Proof. First we prove that if in a space the convolution of two radial kernel functions is a radial kernel function, then the space is harmonic.

In fact, in this case for any $R > 0$ and any smooth kernel function $H(r(x, y))$ the kernel functions

(2.9)
$$H^R(x, y) := \int_{S^{n-1}_{y;R}} H(x, p)\, dp,$$

(2.10)
$$\left.\frac{\partial^2 H^R(x, y)}{\partial R^2}\right|_{R=0} = \frac{(n+1)!}{2n}((-\Delta_y H)(x, y) - \frac{1}{3}\rho(y)H(x, y))$$
$$= \frac{(n+1)!}{2n}\left(H''_x(y) + \frac{\theta'_x(y)}{\theta_x(y)}H'_x(y) - \frac{1}{3}\rho(y)H(x, y)\right)$$

are radial, where Δ_y means the Laplacian acting on the second component, and $\rho(y)$ is the Riemannian curvature scalar. This is possible, if ρ is constant and the Minkowskian mean curvature $\sigma_x(y) := \Theta'_x(y)/\Theta_x(y)$ of the geodesics spheres defines a radial function around x. From this the harmonicity follows.

For the proof of the converse statement we consider first a simply connected compact harmonic manifold and two radial kernel functions $h(r(x, y)$; $g(r(x, y))$ on it. For a fixed point x the eigenfunctions $\varphi_x^{\lambda_i}$ form an orthogonal basis among the radial L^2-functions around x, so h_x can be written as L^2-series

(2.11)
$$h_x = \sum_i \alpha_i \varphi_x^{\lambda_i}.$$

Thus from (2.7) we get

(2.12)
$$\begin{aligned}\int h_x(z) g_y(z)\, dz &= \sum_i \alpha_i \int \varphi_x^{\lambda_i}(z) g_y(z)\, dz \\ &= \sum_i \alpha_i \int_{S^{n-1}_y} \int_0^{R_g} \varphi_x^{\lambda_i}(r_y, \varphi) g(r)\Theta(r) dr d\varphi \\ &= \Omega_{n-1} \sum_i \alpha_i \int_0^{R_g} (A_y \varphi_x^{\lambda_i})(r) g(r)\Theta(r)\, dr \\ &= \Omega_{n-1} \sum_i \alpha_i \left(\int_0^{R_g} \varphi^{\lambda_i}(r) g(r)\Theta(r)\, dr\right) \varphi_x^{\lambda_i}(y),\end{aligned}$$

where Ω_{n-1} denotes the hypersurface area of the euclidean unit sphere S^{n-1}, and R_g is the supporting radius of the function g: $\mathbf{R}_+ \to \mathbf{R}$.

As the functions $\varphi_x^{\lambda_i}$ are radial functions around x, so is the function $(h * g)_x(y) = \int h_x(z) g_y(z)\, dz$. This completely proves the converse statement for the compact case.

In the noncompact case the procedure is similar. In this case for any geodesic ball $B_{x;\delta}$ with the center x and radius $R_h + R_g < \delta$, a system $\{\varphi_x^{\lambda_i}\}_{i=1}^{\infty}$ of radial eigenfunctions can be chosen in such a way that these functions span the radial L^2 function space defined on $B_{x;\delta}$ around x. Using these functions, the radiality of $(h * g)_x(y)$ follows from the same computation as before.

This proposition has several geometric corollaries supporting the Lichnerowicz conjecture.

Corollary 2.1. *Let $B_{x_0 r_1}$ and $B_{y r_2}$ be geodesic balls in a harmonic manifold. Then the volume* $\mathrm{vol}(B_{x_0 r_1} \cap B_{y,r_2})$, *the hypersurface area* $\mathrm{Area}(B_{x_0 r_1} \cap S_{y,r_2})$ *and in general the integral*

$$(2.13) \qquad \int_{B_{x_0 r_1} \cap S_{y r_2}} f(r_{x_0}(P))\, dS_{y,r_2}(P); \qquad f\colon \mathbf{R}_+ \to \mathbf{R}$$

are constant as y moves along the sphere with center x_0 and radius $R = r_{x_0}(y) = \text{constant}$.

The proof is straightforward by using the characteristic functions of the balls in the above proposition.

Another simple but interesting corollary of the generalized mean value theorem (2.6) is the following.

For a radial kernel function $H(r(x,y))$ with $R_H < \infty$ we define the convolution $H * f$ on the $L^2(M^n)$ function space by

$$(2.14) \qquad H * f(x) = \int H(r(x,y)) f(y)\, dy.$$

For a simply connected complete harmonic space we have

Corollary 2.2. *All the globally defined eigenfunctions φ of the Laplacian (with the eigenvalue λ) are the eigenfunctions of the operator $H*$ with the eigenvalue*

$$(2.15) \qquad \Omega_{n-1} \int_0^{R_h} \varphi^\lambda(r) H(r) \Theta(r)\, dr,$$

where Ω_{n-1} is the area of the euclidean unit sphere S^{n-1}.

The proof easily follows from (2.6) by

$$\int \varphi(z) H_y(z)\, dz = \Omega_{n-1} \int_0^{R_h} \varphi^\lambda(r) H(r) \Theta(r)\, dr \varphi(y).$$

3. A generalization of Besse's nice imbedding

A. L. Besse stated a beautiful theorem in [5], where he constructed isometric imbeddings of compact strongly harmonic manifolds into the euclidean spaces in such a way that the images of the geodesics are congruent screw lines in the euclidean space. For the proof he used the heat kernel of the manifolds considered.

Now we generalize this statement considerably as we construct similar imbeddings of an arbitrary harmonic manifold into the Hilbert space l^2. Our method will be different from Besse's as the heat kernel cannot be used for such general cases. On the other hand, our method gives Besse's result as a special case.

First of all we survey some facts about the screw lines in the Hilbert space l^2. A coherent theory of these curves was given by J. von Neumann and I. J. Schoenberg in [28]. They defined these screw lines in l^2 as the rectifiable continuous curves $\mathbf{r}(s)$, parametrized by the arclength s, for which the distance $\|\mathbf{r}(s_1) - \mathbf{r}(s_2)\|$ in the l^2-space depends only on the arclength $s_1 - s_2$ for any two points $\mathbf{r}(s_1)$ and $\mathbf{r}(s_2)$. They called the function

$$S(s) = \|\mathbf{r}(s_0 + s) - \mathbf{r}(s_0)\|^2 \tag{3.1}$$

the screw function of the screw lines considered, and investigated these functions from the viewpoint of positive definite functions.

We remark that the above notion of the screw lines in the Hilbert space l^2 can easily be traced back to the classical screw-line notion. In fact, let $\mathbf{r}(s) \subset l^2$ be a C^∞ curve in l^2, which is a screw line in the above sense with the screw function $S(s)$. The Frenet frame $f_1(s) = \dot{\mathbf{r}}(s), f_2(s) = \ddot{\mathbf{r}}(s)/|\ddot{\mathbf{r}}(s)|, \cdots$, etc. is defined as usual in the classical case together with the curvatures $K_1 = |\dot{\mathbf{r}}| = 1, K_2 = |\ddot{\mathbf{r}}|, \cdots, K_i, \cdots$, etc.

If we transfer the origin of the space l^2 into $\mathbf{r}(0)$, then by the assumption the function $\langle \mathbf{r}(s), \mathbf{r}(s)\rangle$ is independent of the choice of the origin $s = 0$ on the curve $\mathbf{r}$, and therefore the derivatives of this function at $s = 0$ define constant functions along the curve. From this we shall see that curvatures K_i are constant.

We prove this statement by induction. By the Frenet formulas we get $\langle \mathbf{r}(s), \mathbf{r}(s)\rangle^{(4)}_{s=0} = -2K_2^2$, so K_2 is constant indeed. Assuming that the curvatures $K_1, \cdots, K_{k-1}$ are constant we prove that K_k is also constant.

In fact, by the formulas

$$\begin{aligned}
\mathbf{r}^{(1)} &= \mathbf{f}_1, \\
\mathbf{r}^{(2)} &= K_2\mathbf{f}_2, \\
\mathbf{r}^{(3)} &= K_2K_3\mathbf{f}_3 - K_2^2\mathbf{f}_1, \\
&\vdots \\
\mathbf{r}^{(k-1)} &= K_2K_3\cdots K_{k-1}\mathbf{f}_{k-1} + T_{k-1}(K_1, \cdots, K_{k-2}, \mathbf{f}_{k-3}, \mathbf{f}_{k-5}, \dots),
\end{aligned} \tag{3.2}$$

we find

$$
\begin{aligned}
\mathbf{r}^{(k)} &= K_2K_3\cdots K_k\mathbf{f}_k + T_k(K_1,\cdots,K_{k-1},\mathbf{f}_{k-2},\mathbf{f}_{k-4},\dots),\\
\mathbf{r}^{(k+1)} &= K_k'K_2\cdots K_{k-1}\mathbf{f}_k + K_2\cdots K_kK_{k+1}\mathbf{f}_{k+1} - K_2\cdots K_{k-1}K_k^2\mathbf{f}_{k-1}\\
&\quad + T_{k+1}(K_1,\cdots,K_{k-1},\mathbf{f}_1,\cdots,\mathbf{f}_{k-1}),\\
&\quad\vdots\\
\mathbf{r}^{(k+l)} &= T^*_{k+l}(K_i^{(p)},K_i,\mathbf{f}_{k+l},\cdots,\mathbf{f}_{k-l+1})\\
&\quad + (-1)^lK_2\cdots K_{k-l}K_{k-l+1}^2\cdots K_k^2\mathbf{f}_{k-l}\\
&\quad + T_{k+l}(K_1,\cdots,K_{k-l},\cdots,\mathbf{f}_{k-l}),
\end{aligned}
\tag{3.3}
$$

where the terms T_i and T_i^* are suitable functions (linear combinations) of the arguments. Thus for the derivative $\langle\mathbf{r},\mathbf{r}\rangle_{s=0}^{(2k)}$ we have

$$
\begin{aligned}
\langle\mathbf{r},\mathbf{r}\rangle_{s=0}^{(2k)} &= \sum_{l=0}^{k} 2\binom{2k}{l}\langle r^{(l)}, r^{(2k-l)}\rangle_{s=0}\\
&= \sum_{l=0}^{k} 2\binom{2k}{k-l}\langle r^{(k-l)}, r^{k+l}\rangle_{s=0}\\
&= 2(-1)^{k+1}K_2^2\cdots K_k^2 + \sum_{l=0}^{k} 2\binom{2k}{k-l}\langle \mathbf{r}^{(k-l)}, T_{k+l}\rangle_{s=0},
\end{aligned}
\tag{3.4}
$$

from which K_k = constant follows obviously. So the smooth screw lines have constant curvatures. The converse is also obvious.

Now return to the investigations of von Neumann and Schoenberg, who constructed for any screw line $\mathbf{r}(s)$ a continuous one-parameter family U_s of unitary transformations in the space l^2 such that $\mathbf{r}(s)$ is the orbit of U_s of the form

$$\mathbf{r}(s) = U_s(\mathbf{r}(0)).$$

It can also be proved that for two screw lines $\mathbf{r}_1(s)$, $\mathbf{r}_2(s)$ with the same screw function $S_1(s) = S_2(s)$, between the spaces v_i, spanned by $\{\mathbf{r}_i(s)\}_{s\in\mathbb{R}}$, there exists an isometry $v\colon v_1 \to v_2$ which takes $\mathbf{r}_1$ onto $\mathbf{r}_2$.

After this introduction we construct an isometric immersion of a complete simply connected locally harmonic manifold M^n into the Hilbert space $L^2(M^n) \cong l^2$. We remark that this method also gives local imbeddings for a general harmonic manifold without any topological assumption.

For this construction we consider a function $h\colon \mathbf{R}_+ \to \mathbf{R}$ of class C^1 with $h'(0) = 0$ and compact support whose supporting radius R_h is not greater than the radius i_p of injectivity at any point $p \in M^n$.

In the case $i_p = \infty$ we could consider also a function h for which $\int h^2\Theta\,dt, \int (h')^2\Theta\,dt < \infty$, i.e., $h, h' \in L^2_\Theta$.

With the help of h we define the map

$$\Phi_h\colon M^n \to L^2(M^n) \tag{3.5}$$

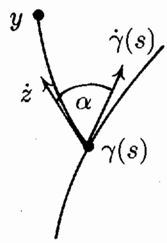
by

$$\Phi_h\colon p\ (\in M^n) \to h_p\ (\in L^2(M^n)), \tag{3.6}$$

where $h_p(y)$ is given by $h_p(y) := h(r(p,y))$ as in §2.

If $\gamma(s)$ is a geodesic of M^n parametrized by the arc length s, then the tangent vectors of $\Phi_h(\gamma(s))$ are functions again. A simple calculation shows

$$\begin{aligned} \frac{d\Phi_h(\gamma(s))}{ds}(y) &= \lim_{t\to 0} \frac{h(r(\gamma(s+t),y)) - h(r(\gamma(s),y))}{t} \\ &= \lim_{t\to 0} \frac{h_y(\gamma(s+t)) - h_y(\gamma(s))}{t} \\ &= -h'(r(\gamma(s),y))\cos\alpha, \end{aligned} \tag{3.7}$$



where α is the angle between $\dot\gamma(s)$ and the tangent vector $\dot z(\gamma(s))$ of the geodesic joining $\gamma(s)$ and y. So these tangent vectors have the constant norm:

$$\|\Phi'_h(\gamma(s))\| = \left[\frac{\Omega_{n-1}}{n}\int_0^{R_h}(h')^2(r)\Theta(r)\right]^{1/2} = \frac{\|h'\|_\Theta}{\sqrt{n/\Omega_{n-1}}}. \tag{3.8}$$

Theorem 3.1(*Imbedding theorem of harmonic spaces*).

(1) *For any radial kernel function $h(r(p,y))$ above the map*

$$\begin{aligned} \mathbf{r}_h = \phi_{qh}\colon M^n &\to L^2(M^n), \\ \mathbf{r}_h\colon p &\to qh_p(y), \end{aligned} \tag{3.9}$$

where $q := \sqrt{n/\Omega_{n-1}}/\|h'\|_\Theta$, is an isometric immersion of a harmonic space M^n into the sphere S_Q of $L^2(M^n)$ with radius $Q = \sqrt{n}\|h\|_\Theta/\|h'\|_\Theta$.

(2) *The geodesics of* $\mathbf{r}_h(M^n)$ *are congruent screw lines in the space* $L^2(M^n) \cong l^2$.

(3) *The submanifold* $\mathbf{r}_h(M^n) \subset S_Q$ *is minimal in the sphere* S_Q *iff the functions* $h_p(y)$ *are eigenfunctions of the Laplacian* Δ. *In this case the eigenvalues have the form* $\lambda = n/Q^2$ *automatically because* $\int \nabla_i h \nabla^i h = -\int h\Delta h$.

Proof. Using (3.8), for any geodesic $\gamma(s)$ of M^n we have $||r_h'(\gamma(s))||_{L^2} = 1$, so $\mathbf{r}_h$ is an isometric immersion indeed. From

$$\int_{M^n} h_p^2 = \Omega_{n-1} \int h^2(r)\Theta(r)\,dr = \Omega_{n-1}||h||_\Theta^2$$

we obviously get

$$\mathbf{r}_h(M^n) \subset S_Q. \tag{3.10}$$

Furthermore for any two points $\mathbf{r}_h(\gamma(s_1))$ and $\mathbf{r}_h(\gamma(s_2))$ the inner product

$$\begin{aligned} F_\gamma(s_1, s_2) &:= \langle \mathbf{r}_h(\gamma(s_1)), \mathbf{r}_h(\gamma(s_2))\rangle \\ &= \frac{n/\Omega_{n-1}}{||h'||_\Theta^2} \int h_{\gamma(s_1)}(y) h_{\gamma(s_2)}(y), \end{aligned} \tag{3.11}$$

and therefore also the function

$$||r_h(\gamma(s_1)) - r_h(\gamma(s_2))||^2 = 2Q^2 - 2F_\gamma(s_1, s_2) = 2Q^2 - 2F(|s_1 - s_2|)$$

depends only on the geodesics distance $|s_1 - s_2|$ by Proposition 2.1. This means that the geodesics of $\mathbf{r}_h(M^n)$ are congruent screw lines in $L^2(M^n)$ with the common screw function $2(Q^2 - F(s)) = S(s)$.

For the proof of the last statement we consider also an orthonormal basis $\varphi_1, \varphi_2, \ldots$ in the Hilbert space $L^2(M^n)$ with the coordinate functions

$$x^i(p) = \langle \varphi_i, \mathbf{r}_h(p)\rangle = \int \varphi_i \mathbf{r}_h(p). \tag{3.12}$$

By a well-known theorem [21, p. 342] the submanifold $\mathbf{r}_h(M_n) \subset S_Q$ is minimal in S_Q if and only if

$$\Delta x^i = (n/Q^2)x^i, \qquad i = 1, 2, \cdots, \tag{3.13}$$

i.e., if and only if

$$\int_{M^n} \varphi_i(x)\Delta_p h(r(p, x))\,dx = \frac{n}{Q^2}\int_{M^n} \varphi_i(x) h(r(p, x))\,dx, \tag{3.14}$$

and consequently

$$\Delta_p h(r(p, x)) = (\Delta_x h_p)(x) = \frac{n}{Q^2} h_p(x) \tag{3.15}$$

is satisfied for any $p \in M^n$. This proves the theorem completely.

In the cases of compact strongly harmonic manifolds the space is a Blaschke manifold with a simply closed geodesic with constant length $2L$ [5]. So for any eigenvalue $\lambda \in \text{spect}\{\lambda_i\}_{M^n}$ of the spectrum a uniquely determined radial eigenfunction φ_x^λ exists with $\varphi_x^\lambda(x) = 1$ and the eigenvalue λ, as the space is globally averageable. The functions $\varphi_x^\lambda(\cdot) = \varphi^\lambda(r(x,\cdot))$ span a finite dimensional subspace in $L^2(M^n)$, namely the eigensubspace V^λ. Thus the map

$$\mathbf{r}_{\varphi^\lambda} : M^n \to L^2(M^n) \tag{3.16}$$

maps the manifold M^n into the sphere S_Q of V^λ such that all the geodesics are congruent screw lines in V^λ. The minimality of $\mathbf{r}_{\varphi^\lambda}(M^n) \subset S_Q$ in S_Q follows from the fact that the φ_x^λ are eigenfunctions for any x. Besse constructed exactly these maps for compact strongly harmonic manifolds, and called them Nice Imbeddings of compact strongly harmonic manifolds.

We describe yet some more useful formulas. Let $\varphi_1, \cdots, \varphi_l$ be an orthonormal basis in V^λ. Then φ_x^λ is of the form

$$\varphi_x^\lambda(y) = a_1\varphi_1(y) + \cdots + a_l\varphi_l(y), \tag{3.17}$$

with

$$a_i(x) = \int_{M^n} \varphi_x^\lambda(y)\varphi_i(y)\,dy = \Omega_{n-1}\varphi_i(x)\int_0^L (\varphi^\lambda(r))^2\Theta(r)\,dr. \tag{3.18}$$

Thus for any strongly harmonic manifold, we have

$$\begin{aligned} \varphi_x^\lambda(y) &= \Omega_{n-1}\int_0^L (\varphi^\lambda(r))^2\Theta(r)\,dr \sum_{i=1}^{l} \varphi_i(x)\varphi_i(y) \\ &= A_\lambda \sum \varphi_i(x)\varphi_i(y). \end{aligned} \tag{3.19}$$

From these we get

$$\langle \varphi_{x_1}^\lambda, \varphi_{x_2}^\lambda \rangle = A_\lambda^2 \sum \varphi_i(x_1)\varphi_i(x_2) = A_\lambda \varphi_{x_1}^\lambda(x_2), \tag{3.20}$$

which means that the restriction of the eigenfunction φ_x^λ onto a geodesic $\gamma(r)$ with $\gamma(0) = x$ is of the form

$$\varphi^\lambda(r) = \varphi_x^\lambda(\gamma(r)) = B_\lambda \langle \mathbf{r}_{\varphi^\lambda}(x), \mathbf{r}_{\varphi^\lambda}(\gamma(r)) \rangle, \tag{3.21}$$

where B_λ obviously depends only on λ.

4. The proof of Lichnerowicz's conjecture for compact simply connected harmonic manifolds

We prove the conjecture for compact simply connected harmonic manifolds step-by-step using more lemmas. Note that then the conjecture is established for a compact harmonic manifold with finite fundamental group.

First of all we answer the following elementary question. Let $f_h(t) := f(t+h)$ stand for the parallel displacement of a function $f\colon \mathbf{R} \to \mathbf{R}$ with respect to a real number $h \in \mathbf{R}$. Our question is as follows: What are the continuous functions $f\colon \mathbf{R} \to \mathbf{R}$ for which the functions $\{f_h\}_{h\in\mathbb{R}}$ span a function space of finite dimension? Although the following answer is classical, we will give a short proof here, for the sake of completeness.

Lemma 4.1. *The functions $\{f_h\}_{h\in\mathbb{R}}$ span a function-space V of finite dimension iff f is of the form*

$$(4.1)\qquad f(x) = \sum_{i=1}^{k} P_i(x)\sin\alpha_i x + Q_i(x)\cos\beta_i x + R_i(x)e^{\gamma_i x},$$

where $P_i(x)$, $Q_i(x)$, and $R_i(x)$ are polynomials.

Proof. It is easy to show that for functions of the form (4.1) the function-space V spanned by $\{f_n\}_{h\in\mathbb{R}}$ is of finite dimension indeed.

Conversely, if V is of finite dimension, then let

$$(4.2)\qquad \Phi_h\colon V \to V, \qquad \Phi_h\colon g(x) \to g_h(x)$$

be the operator of the parallel displacement in V. Thus $\{\Phi_h\}_{h\in\mathbb{R}}$ is a continuous one-parameter family of linear transformations in V, since

$$(4.3)\qquad \begin{aligned}&\Phi_h(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1\Phi_h(g_1) + \alpha_2\Phi_h(g_2);\ \alpha_1,\alpha_2 \in \mathbf{R},\ g_1, g_2 \in V,\\ &\Phi_0 = \mathrm{id}, \qquad \Phi_{h_1+h_2} = \Phi_{h_1}\circ\Phi_{h_2}\end{aligned}$$

hold trivially. By the Cartan theorem (which is the finite dimensional version of the Stone theorem), Φ_h is of the form

$$(4.4)\qquad \Phi_h = \exp hX = \sum \frac{h^k}{k!}X^k$$

for a linear endomorphism $X\colon V \to V$. So the function $f(x)$ is not only continuous but also of class C^∞ for which the ith derivative is just the continuous function $X^i(f)$. More precisely, f is an analytic function, as the curve $c(h)\colon h \to f_h = \Phi_h(f)$ in V is analytic with the convergent Taylor expansion

$$(4.5)\qquad f_h(x) = \sum \frac{h^k}{k!}X^k(f)/k, \qquad |h| < \varepsilon < 0.$$

Therefore the Taylor expansion

$$f(x) = \sum \frac{f^{(i)}(0)}{i!} x^i = \sum \frac{x^i}{i!} X^i(f)(0) \tag{4.6}$$

is convergent for $|x| < \varepsilon$.

It is also obvious that the derivatives $d^i f/dx^i := f^{(i)}$ belong to V, and as V is of finite dimension, $f^{(k)}$ is a linear combination of the functions $f^{(0)} = f, f^{(1)}, \cdots, f^{(k-1)}$ for some k. Therefore the function f is the solution of a differential equation of constant coefficients of the form

$$\sum_{i=0}^{k} A_i f^{(i)} = 0, \qquad A_i \in \mathbf{R}, \quad A_k = 1, \tag{4.7}$$

and f is of the form (4.1) by a rather well-known classical theorem. q.e.d.

Using Allamigeon's theorem, we assume that the space is normalized in such a way that the total length of a geodesic is 2π. So the generator function $\varphi^\lambda(r)$ of a radial eigenfunction with $\lambda \in \mathrm{Spect}\{\lambda_i\}_{M^n}$ is a function with period 2π.

Lemma 4.2. *The functions $\varphi^\lambda(r)$, $\lambda \in \{\lambda_i\}_{M^n}$, of a normalized harmonic manifold with the diameter π are of the form $\varphi^\lambda(r) = P_\lambda(\cos r)$, where the P_λ's are polynomials.*

Proof. Since the functions $\varphi^\lambda_x \in V^\lambda$ span the finite dimensional eigen-subspace V^λ, for any geodesic $\gamma(r)$ the functions $\varphi^\lambda_{\gamma(r)}$ span a finite dimensional space. The restrictions of the functions $\varphi^\lambda_{\gamma(r)}$ to γ form a parallel displaced family of functions in the above sense by Lemma 4.1. As these functions span a finite dimensional function-space and are even periodic functions, the generator function $\varphi^\lambda(r)$ is of the form

$$\varphi^\lambda(r) = \sum_{i=1}^{k} A_i \cos \alpha_i r, \qquad A_i, \alpha_i \in \mathbf{R}. \tag{4.8}$$

We prove that the distinct (!) values α_i are uniquely determined natural numbers.

The distinct values α_i are uniquely determined for φ^λ. Supposing the contrary we have a nontrivial linear combination

$$\sum_{i=1}^{l} B_i \cos \alpha_i r = 0. \tag{4.9}$$

By the derivation we have

$$\sum_{i=1}^{l} B_i \alpha_i^{2k} = 0, \qquad k = 0, 1, 2, \cdots, \tag{4.10}$$

which is a contradiction, since the Vandermonde matrix $\{a_i^k\}_{i=1,\cdots,l}^{k=0,\cdots,l-1} := \{\alpha_i^{2k}\}$ has nonvanishing determinant.

From the periodicity $\varphi^\lambda(r+2\pi) = \varphi^\lambda(r)$ and the above consideration we get $\cos 2\alpha_i\pi = 1$ and $\sin 2\alpha_i\pi = 0$. So any value α_i in (4.8) is a natural number and therefore, by the Chebyshev polynomials, $\varphi^\lambda(r)$ is a polynomial of $\cos r$. (This lemma can also be proved by using Fourier series.) q.e.d.

In the following we prove a similar statement for the density function $\Theta^2(r)$.

Lemma 4.3. *The function* $\Theta^2(r)$ *is also a trigonometric polynomial of the form* $\Theta^2(r) = T(\cos r)$ *for any compact normalized harmonic manifold.*

Proof. Let

$$\mathbf{r}_\lambda : M^n \to V^\lambda \tag{4.11}$$

be the Nice Imbedding of M^n into V^λ with respect to an eigenvalue $\lambda \in \{\lambda_i\}_{M^n}$. We consider a variation x_r^s, $-\varepsilon < s < \varepsilon$, of a geodesic $x_r = x_r^0$. Then the map

$$\mathbf{r}_\lambda(r,s) := \mathbf{r}_\lambda(x_r^s) : \mathbb{R} \times (-\varepsilon, \varepsilon) \to V^\lambda \tag{4.12}$$

has the property that for any values of s the curves $r \to \mathbf{r}_\lambda(r,s)$ are congruent screw lines in V^λ. So a differential operator

$$L = \sum_{i=0}^{k} A_i \frac{d^i}{dr^i}, \qquad A_i \in \mathbf{R}, \tag{4.13}$$

of constant coefficients exists[1] such that

$$L(\mathbf{r}_\lambda) = 0 \tag{4.14}$$

holds for any point (r,s). Thus we get

$$0 = \frac{\partial}{\partial s} L(\mathbf{r}_\lambda) = L\left(\frac{\partial \mathbf{r}_\lambda}{\partial s}\right), \tag{4.15}$$

which means that the Jacobian field

$$Y(r) := \left.\frac{\partial \mathbf{r}_\lambda}{\partial s}\right|_{s=0} \tag{4.16}$$

is also a solution of the differential equation

$$L(Y) = 0. \tag{4.17}$$

[1]This can be derived by using the last Frenet formula and also formulas (3.2) for the expression of the Frenet basis $\{\mathrm{f}_i\}$ with the help of $\mathbf{r}^{(k)} - s$.

Let $e_1, \cdots, e_p$ be an orthonormal basis in V^λ. Since the differential equations $L(Y^i) = 0$ are satisfied for the functions $Y^i(r) := \langle Y(r), e_i \rangle$, as in the previous lemma, Y^i is a trigonometric polynomial of the form

$$Y^i(r) = \sum_{j=0}^{k_i} A_j \sin jx + B_j \cos jx. \tag{4.18}$$

Now let $Y_{(1)}, \cdots, Y_{(n-1)}$ be Jacobian fields along x_r with $Y_{(j)}(0) = 0$; furthermore the vectors $E_{(j)} := Y'_{(j)}(0)$ form an orthonormal basis in the hyperspace of $T_{x_0}(M^n)$ orthogonal to $\dot{x}_0$. So the norm of the $(n-1)$-form

$$Y_{(1)} \wedge Y_{(2)} \wedge \cdots \wedge Y_{(n-1)}(r) \tag{4.19}$$

along x_r is just $\Theta(r)$.

On the other hand we have

(4.20)

$$\begin{aligned} Y_{(1)} \wedge \cdots \wedge Y_{(n-1)}(r) &= \sum_{1 \le j_1, \cdots, j_{n-1} \le p} Y^{j_1}_{(1)}(r) \cdots Y^{j_{n-1}}_{(n-1)} e_{j_1} \wedge \cdots \wedge e_{j_{n-1}} \\ &= \sum_{1 \le i_1 < \cdots < i_{n-1} \le p} Q^{i_1 i_2 \cdots i_{n-1}}(Y^l_{(k)}) e_{i_1} \wedge \cdots \wedge e_{i_{n-1}}, \end{aligned}$$

where the functions $Q^{i_1, \cdots, i_{n-1}}$ are suitable polynomials of the functions $Y^l_{(k)}$, i.e., these coefficients are trigonometric polynomials again. Thus the function

$$\Theta^2 = \sum (Q^{i_1 \cdots i_{n-1}})^2 \tag{4.21}$$

is a trigonometric polynomial. As $\Theta^2(r)$ is a periodic even function, it is of the form $\Theta^2(r) = T(\cos r)$, where $T(x)$ is a polynomial. q.e.d.

Now we examine the roots of the polynomials P_λ and T. Our aim is to prove that the polynomial T has only the roots $+1$ and -1. First of all we consider the polynomial P_λ.

Lemma 4.4. *Neither $+1$ nor -1 is a root of P_λ; furthermore all the roots of P_λ have multiplicity one.*

Proof. Since $\varphi_\lambda(r) = P_\lambda(\cos r)$, $\varphi_\lambda(0) = P_\lambda(1) = 1$ holds, the value $+1$ is not a root of P_λ. Let us introduce also the function $z(r) := \varphi_\lambda(\pi - r)$, for which we have

$$z'' + \tilde{\sigma} z' = -\lambda z, \tag{4.22}$$

where $\tilde{\sigma}(r) := -\sigma(\pi - r)$ is a positive function for small values of r, as the function σ is negative near π. (In fact, the $\Theta(r)$ is a decreasing function near π.) As $z'(0) = -\varphi'_\lambda(\pi) = \sin(\pi) P'(\cos(\pi)) = 0$, we have $z(0) =$

$\varphi_\lambda(\pi) = P_\lambda(\cos\pi) = P_\lambda(-1) \neq 0$ by virtue of Lemma 2.1. Thus the value -1 is not a root of P_λ.

Now we return to the second part of the lemma. The equation

$$\varphi_\lambda'' + \frac{\Theta'}{\Theta}\varphi_\lambda' = -\lambda\varphi_\lambda \tag{4.23}$$

can also be written in the form:

$$\begin{aligned}\frac{\Theta'}{\Theta} + \frac{\varphi_\lambda''}{\varphi_\lambda'} &= \frac{1}{2}\left(\frac{(\Theta^2)'}{\Theta^2} + \frac{((\varphi_\lambda')^2)'}{(\varphi_\lambda')^2}\right)\\ &= \frac{1}{2}\frac{(\Theta^2(\varphi_\lambda')^2)'}{\Theta^2(\varphi_\lambda')^2} = -\lambda\frac{\varphi_\lambda}{\varphi_\lambda'}.\end{aligned} \tag{4.24}$$

The function $\Theta^2(\varphi_\lambda')^2$ is a trigonometric polynomial of the form $\Theta^2(r)(\varphi_\lambda')^2(r) = Q(\cos r)$ by the above lemmas, therefore

$$(\ln Q(\cos r))' = 2\lambda\frac{P_\lambda(\cos r)}{\sin rP'(\cos r)}, \tag{4.25}$$

$$\ln Q(\cos r) = -2\lambda\int\frac{P_\lambda(\cos r)}{-\sin rP'(\cos r)}\,dr. \tag{4.26}$$

Using the substitution $x = \cos r$ we get

$$\ln Q(x) = -2\lambda\int\frac{P_\lambda(x)}{(1-x^2)P'(x)}\,dx. \tag{4.27}$$

Let $K_1, \cdots, K_r$ be the roots of P_λ with respective multiplicities $a_1, \cdots, a_r$. Then the derived polynomial P_λ' has values K_i as roots exactly with multiplicities $(a_i - 1)$. Furthermore for P_λ' we have additional new roots $\mu_1, \cdots, \mu_l$ (different from the K_i) with the respective multiplicities, say, $b_1, b_2, \cdots, b_l$. So we have

$$\ln Q(x) = \int q\frac{(x-K_1)\cdots(x-K_r)}{(1-x)(1+x)(x-\mu_1)^{b_1}(x-\mu_2)^{b_2}\cdots(x-\mu_l)^{b_l}}\,dx, \tag{4.28}$$

where $q = -2\lambda/(a_1 + \cdots + a_r)$ is a constant.

Using the method of the partial fraction for the integration of the right side, we have that this integral is of the form $\ln Q(x)$ for a polynomial $Q(x)$ if and only if $b_1 = b_2 = \cdots = b_l = 1$ and $\mu_i \neq \pm 1$. Furthermore in this case $Q(x)$ is of the form

$$Q(x) = a(1-x)^A(1+x)^B(x-\mu_1)^{B_1}\cdots(x-\mu_l)^{B_l} \tag{4.29}$$

with suitable constants $a, A, B, B_1, \cdots, B_l$.

On the other side we have

$$\frac{(\Theta^2)'}{\Theta^2} = -\frac{((\varphi_\lambda')^2)'}{(\varphi_\lambda')^2} + \frac{(Q(\cos r))'}{Q(\cos r)}, \tag{4.30}$$

$$T(\cos r) = (\varphi_\lambda')^{-2}(r)Q(\cos r) = (1-\cos^2 r)^{-1}(P'(\cos r))^{-2}Q(\cos r),$$

and so

$$\begin{aligned} T(x) = p(1-x)^{A-1}(1+x)^{B-1}(x-K_1)^{-2(a_1-1)} \\ \cdots(x-K_r)^{-2(a_r-1)}(x-\mu_1)^{B_1-2}\cdots(x-\mu_l)^{B_l-2}, \end{aligned} \tag{4.31}$$

where $K_i \neq \mu_j$ and $K_i \neq \pm 1$. So, if some multiplicity a_i were greater than 1, then $-2(a_i - 1) < 0$ and thus $T(x)$ would not be a polynomial. This proves completely the remaining statement

$$a_1 = a_2 = \cdots = a_r = 1.$$

Now we turn to the examination of the roots of the polynomial $T(x)$. The values $+1$ and -1 are roots of T as the $\Theta^2(r) = T(\cos r)$ vanishes at $r = 0$ and at $r = \pi$. The multiplicity of these roots is denoted by A (resp. B).

Let $\gamma_1, \cdots, \gamma_l$ be the other roots of $T(x)$ with the respective multiplicities $G_1, \ldots, G_l$. So $\Theta^2(r)$ is of the form

$$\begin{aligned} \Theta^2(r) &= c(1-\cos r)^A(\cos r+1)^B(\cos r-\gamma_1)^{G_1}\cdots(\cos r-\gamma_l)^{G_l}, \\ \Theta^2(r) &= c\sin^p r(1-\cos r)^q(\cos r-\gamma_1)^{G_1}\cdots(\cos r-\gamma_l)^{G_l}, \end{aligned} \tag{4.32}$$

with $p = 2A$ and $q = B - A$.

Lemma 4.5. *All the roots $\gamma_i \neq \pm 1$ of $T(x)$ are also the roots of the polynomial $P_\lambda'(x)$, $\lambda \in \{\lambda_i\}_{M^n}$.*

Proof. From the equation

$$\frac{1}{2}\frac{(\Theta^2)'}{\Theta^2}\varphi_\lambda' = -\varphi_\lambda'' - \lambda\varphi_\lambda \tag{4.33}$$

we have

$$\frac{1}{2}(1-x^2)\frac{T'(x)}{T(x)}P'(x) = -\lambda P(x) + xP'(x) - (1-x^2)P''(x), \tag{4.34}$$

and thus from (4.31) the function

$$\frac{1}{2}(1-x^2)P_\lambda'(x)\left(\frac{-A}{1-x} + \frac{B}{(1+x)} + \frac{G}{(x-\gamma_1)} + \cdots + \frac{G_l}{(x-\gamma_l)}\right) \tag{4.35}$$

is a polynomial. This is possible if and only if the roots γ_i are also the roots of $P_\lambda'(x)$. q.e.d.

The following lemma is much more important for these considerations.

Lemma 4.6. *All the roots* $K_1, \cdots, K_r$ *of* P_λ *and all the roots* $\mu_1, \cdots, \mu_{r-1}$ *of* P'_λ *are real numbers lying in the interval* $(-1, 1)$, *i.e.,*

(4.36) $$-1 < K_1 < \mu_1 < K_2 < \mu_2 < \cdots < \mu_{r-1} < K_r < 1.$$

Proof. By formula (4.24) we have

(4.37) $$(\Theta^2(\varphi'_\lambda)^2)' = -2\lambda\Theta^2\varphi_\lambda\varphi'_\lambda,$$

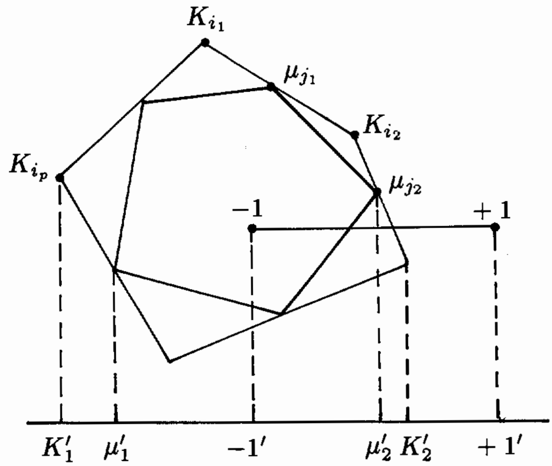
and so

(4.38) $$((1-x^2)T(x)P'(x)P'(x))' = -2\lambda T(x)P(x)P'(x).$$

The roots of the polynomial $(1-x^2)T(x)P'(x)P'(x)$ are exactly the values $+1, -1, \mu_1, \cdots, \mu_{r-1}$, and the roots of $T(x)P(x)P'(x)$ are exactly the values $+1, -1, K_1, \cdots, K_r, \mu_1, \cdots, \mu_{r-1}$ by the above lemmas. As the roots of a derived polynomial lie in the convex hull of the roots of the original polynomial by Lucas' theorem, we have

(4.39) $$\{+1, -1, K_1, \cdots, K_r, \mu_1, \cdots, \mu_{r-1}\} \subset \text{conv}\{+1, -1, \mu_1, \cdots, \mu_{r-1}\}.$$

We show that this situation is possible only in the case where all the roots $K_1, \cdots, K_r$ of P_λ (and consequently also all the roots $\mu_1, \cdots, \mu_r$ of P'_λ) lie in the interval $(-1, +1)$. In fact, the convex hull of the roots $K_1, \cdots, K_r$ of P_λ also contains the roots $\mu_1, \cdots, \mu_{r-1}$ of P'_λ. As the multiplicity of any root K_i is exactly one, the vertices $K_{i_1}, \cdots, K_{i_l}$ of $\text{conv}\{K_1, \cdots, K_r\}$ are different from the vertices $\mu_{j_l}, \cdots, \mu_{j_k}$ of $\text{conv}\{\mu_1, \cdots, \mu_{j_k}\}$. So if g is such a line on the complex plane, which is not orthogonal to any of the sides of $\text{conv}\{K_1, \cdots, K_r\}$, then the orthogonal projection of $\text{conv}\{K_1, \cdots, K_r\}$ onto g is an interval $[K'_1, K'_2]$ which *properly* contains the orthogonal projection $[\mu'_1, \mu'_2]$ of $\text{conv}\{\mu_1, \cdots, \mu_{r-1}\}$, i.e., in which $K'_1 < \mu'_1 < \mu'_2 < K'_2$ holds.



Now, if the roots $K_1, \cdots, K_r$ did not lie in the interval $(-1, +1)$, then it would be possible to choose such a line g which has the following additional property: The orthogonal projection $[-1', +1']$ of $[-1, +1]$ onto g does not contain the orthogonal projection $[K_1' K_2']$ of $\mathrm{conv}\{K_1, \cdots, K_r\}$. So at least one of the points K_1', K_2' (say K_1') is not contained in $[-1', +1']$. In this case we have

$$K_1' \notin \mathrm{conv}\{\mu_1', \mu_2', -1', 1'\}, \quad K_{i_p} \notin \mathrm{conv}\{+1, -1, \mu_1, \cdots, \mu_{r-1}\}, \tag{4.40}$$

where K_{i_p} is the root of P_λ, whose orthogonal projection onto g is just the point K_1'. This contradicts the property (4.39), so all the roots $K_1, \cdots, K_r$ are contained in $(-1, +1)$ indeed. The arrangement (4.36) of the roots follows immediately from the fact that the multiplicity of any root K_i is one. q.e.d.

Now we return to the roots of the polynomial $T(x)$.

Lemma 4.7. *The polynomial $T(x)$ has only the roots $+1$, -1, so the density function $\Theta(r)$ of a compact normalized harmonic manifold is of the form*

$$\Theta(r) = D \ \sin^p(r)(1 - \cos r)^q. \tag{4.41}$$

Proof. If $T(x)$ had a root μ different from ± 1, then μ would be the root also of $P_\lambda'(x)$ by Lemma 4.5. Using Lemma 4.6, μ would be real with $-1 < \mu < 1$. So if $0 < r_0 < \pi$ were the value for which $\cos r_0 = \mu$ holds, then we would have $\Theta^2(r_0) = T(\cos r_0) = T(\mu) = 0$, which is a contradiction, as $\Theta^2(r)$ is strictly positive on the interval $0 < r < \pi$ and vanishes only at the endpoints 0 and π. So $T(x)$ has only the roots ± 1 and $\Theta(r)$ is of the form

$$\Theta(r) = D(1 - \cos r)^{A^*}(1 + \cos r)^B = D \sin^p r(1 - \cos r)^q, \tag{4.42}$$

where $p = 2B^*$ and $q = A^* - B^*$.

Remark. In the case of two-point homogeneous spaces we have S^n: $\Theta = D \sin^{n-1} r$; $\mathbf{P}^n(\mathbf{C})$: $\Theta = D \sin r(1 - \cos r)^{(n-2)/2}$; $P^n(\mathbf{H})$: $\Theta = D \sin^3 r(1 - \cos r)^{(n-4)/2}$ $P^{16}(Cay)$: $\Theta = D \sin^7 r(1 - \cos r)^4$. It should be remarked that these are the only possibilities for a compact strong harmonic manifold. In fact from (4.41) and $\Theta(r) = r^{n-1} + \{\text{higher order terms}\}$ we get: $p + 2q = n - 1$; $D = 2^{2q} = 4^q$. On the other hand the Bott-Samelson Theorem [5] (which describes the cut locus of a Blaschke manifold) implies $2q = \dim$ (cut locus) $= 0$; $n - 2$; $n - 4$ or 8 and in the last case $n = 16$. (We do not use this remark in the following considerations.)

Lemma 4.8. *Any normalized* ($2L = 2\pi$) *compact strongly harmonic manifold has a Laplacian eigenfunction of the form*

$$\varphi_\lambda = B\cos r + A, \qquad A + B = 1, \tag{4.43}$$

whose eigenvalue λ *is the least nontrivial eigenvalue of the Laplacian. The spectrum* (*without multiplicity!*) *is*: $\{\lambda_n = n(n+p+q)\}_{n\in\mathbb{N}}$.

Proof. From $\Theta = D\sin^p r(1-\cos r)^q$ we have

$$\frac{\Theta'}{\Theta} = \frac{p\cos r}{\sin r} + \frac{q\sin r}{1-\cos r} = \frac{(p+q)\cos r + q}{\sin r}, \tag{4.44}$$

and therefore, for the function $u = \cos r + (q/(p+q+1))$,

$$u'' + \frac{\Theta'}{\Theta}u' = -\cos r - ((p+q)\cos r + q) = -(p+q+1)u, \tag{4.45}$$

i.e., the function $u = \cos r + (q/(p+q+1))$ is an eigenfunction with the eigenvalue $\lambda = p+q+1$.

It can be seen easily that for any $n \in \mathbf{R}_+$ an eigenfunction of the form

$$\cos^n r + A_1\cos^{n-1} r + \cdots + A_{n-1}\cos r + A_n, \qquad A_i \in \mathbf{R}, \tag{4.46}$$

exists, and its eigenvalue is

$$\lambda_n = n(n+p+q), \tag{4.47}$$

which proves the lemma completely.

Lemma 4.9. *Let* $\mathbf{r}\colon M^n \to V^{\lambda_1}$ *be the Nice Imbedding of a compact normalized harmonic manifold with respect to the first nontrivial eigenfunction* $\cos r + A$. *Then the geodesics of* $\mathbf{r}(M^n)$ *are circles of radius* 1 *in* V^{λ_1}.

Proof. Let $\mathbf{r}(r) = \mathbf{r}(\gamma(r))$ be the image set of a geodesic $\gamma(r)$. Then by formula (3.21) and Lemma 4.8 the function $\langle \mathbf{r}(0), \mathbf{r}(r)\rangle$ is of the form $B\cos r + A\cdot B$, $A, B \in \mathbf{R}$, so for any r we get

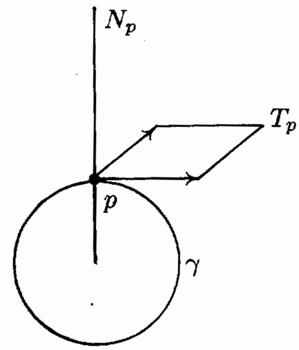
$$\langle \mathbf{r}(0), \mathbf{r}''' + r'\rangle = 0. \tag{4.48}$$

As $\mathbf{r}(0)$ is arbitrary on the geodesic $\mathbf{r}(\gamma)$, and the vectors $\mathbf{r}''' + \mathbf{r}'$ lie in the subspace spanned by the vectors $\{\mathbf{r}(\gamma)\}$, we get

$$\mathbf{r}''' + \mathbf{r}' = 0. \tag{4.49}$$

From the Frenet formulas it follows that $\mathbf{r}(\gamma)$ is a plane curve of constant curvature $+1$, i.e., it is a circle. q.e.d.

The following lemma completely proves the conjecture for the compact harmonic manifolds with finite fundamental groups.



Lemma 4.10. *Let $M^n \subseteq \mathbf{R}^{k+n}$ be a submanifold such that all the geodesics of M^n are circles in $\mathbf{R}^{k+n}$. Then M^n is a symmetric space and is further a two-point homogeneous space.*

Proof. Let N_p be the orthogonal complement of the tangent space $T_p(M^n)$ at a point $p \in M^n$ in $\mathbf{R}^{n+k}$, and let

$$\tau_p \colon \mathbf{R}^{n+k} \to \mathbf{R}^{n+k} \tag{4.50}$$

be the reflexion with respect to the subspace N_p. Then τ_p is an isometry of the euclidean space $\mathbf{R}^{n+k}$.

As the curvature vectors $\mathbf{r}_p''$ of the geodesics through p lie in N_p, τ_p leaves these geodesics together with the whole submanifold M^n invariant. Thus the τ_p induces an isometry on M^n, which is obviously the geodesics involution at p. So M^n is a symmetric space. It has the rank one, because all the other symmetric spaces have nonclosed geodesics on a maximal torus. This proves the Lemma and the conjecture completely.

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